Vector statistics of correlated Gaussian fields

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Vector statistical distribution functions are derived for correlated Gaussian fields. The results may be used to describe the interesting cases of polarization of multiply scattered optical waves from disordered media. The dependence of the parameters (the elements of the covariant Hermitian matrix), which characterize the statistics on the incident polarization field, on the incident and scattered wave-vector direction, and on the medium (with slab geometry) are also given.

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I. INTRODUCTION

In recent years, there has been a resurgence of interest in multiple scattering of light in disordered media. A scalar wave model for the optical field was successfully used in many instances for spatial [1], angular [2, 3], wavelength [4–6] or time correlation in dynamic systems [7–9], weak localization [10–17] and antilocalization [18], the optical memory effect [2, 19–21], its time-reversed counterpart [22–24], and image reconstructions [25–27]. Recently, it is becoming increasingly apparent, however, that the vector nature of light plays an important, indeed sometimes dominant, role in many diverse phenomena [10, 11, 23, 27–34]. Scattering of (partially) coherent light in highly random media generally produces a partially polarized speckle pattern. The degree of polarization of the scattered light and other vector statistical quantities depend on the details of the random medium, the polarization state of the incident light and its degree of coherence, and the incident and scattered wave-vector directions.

Recently, the statistical properties of temporally random (partially polarized) light [28, 35, 36] and also of spatially random multiply scattered waves [29, 30, 37–41] from random media (speckle patterns) have attracted much theoretical and experimental interest. The intensity (I) probability distribution function (PDF) for a multiply scattered scalar field has been long known to be the Rayleigh distribution $P(I) = \langle I\rangle^{-1}\ \exp(-I/\langle I\rangle)$, where $\langle I\rangle$ is the ensemble-average intensity. Vector statistics are characterized by the probabilities of the change in polarization state for different Feynman paths which are each built from many single-scattering events that rotate the incident polarized field randomly. Thus, an important feature of multiple scattering is to change the incident polarization state in a statistical fashion.

The statistics of speckle patterns (or temporal randomness) are related to the covariance matrix $\tilde{\mathbf{J}}$ of the scattered light, which depends both upon the properties of the random medium and the incident polarization state. In this paper, results are presented for the vector statistics of multiply scattered light based upon explicit calculations for the covariance matrix.

The scattered polarization state can be described by groups of three or four variables. First, some of the most interesting vector variables and the connections between them are presented. Then, the joint and marginal probabilities of these variables are given, assuming correlated Gaussian fields. Finally the parameters that characterize the statistics (the elements of the covariance $\tilde{\mathbf{J}}$ matrix) are developed in terms of the system properties and incident polarization state.

II. THE VECTOR VARIABLES

For a plane wave propagating normal to the x-y plane, the basic variables are the real and imaginary parts of the fields $E_x = E_x^r + iE_x^i$ and $E_y = E_y^r + iE_y^i$, where $-\infty < E_{x,y}^{r,i} < \infty$. The amplitudes ($A$) and phase ($\delta$) are

$$A_x = \sqrt{I_x} = \sqrt{(E_x^r)^2 + (E_x^i)^2},\quad 0 \le A_x < \infty \tag{1a}$$

$$A_y = \sqrt{I_y} = \sqrt{(E_y^r)^2 + (E_y^i)^2},\quad 0 \le A_y < \infty \tag{1b}$$

$$\delta_x = \arg(E_x),\quad 0 \le \delta_x < 2\pi \tag{1c}$$

$$\delta = \arg(E_y) - \arg(E_x),\quad -2\pi \le \delta < 2\pi\ . \tag{1d}$$

At each point in the scattered field, the polarization is generally elliptical, with the ellipse variables being given in terms of A_x, A_y, δ (δ_x does not affect the shape of this polarization ellipse, but does determine the initial conditions for the electric-field amplitude and direction of the principal axes). The amplitudes of the major (A_a) and minor (A_b) axes, and the corresponding intensities $I_a = (A_a)^2,\ \ I_b = (A_b)^2$ are

$$I_a = (A_a)^2 = \tfrac{1}{2}\{(A_x)^2 + (A_y)^2 + \sqrt{[(A_x)^2 - (A_y)^2]^2 + 4(A_x)^2(A_y)^2\cos^2\delta}\}\ , \tag{2a}$$

$$I_b = (A_b)^2 = \tfrac{1}{2}\{(A_x)^2 + (A_y)^2 - \sqrt{[(A_x)^2 - (A_y)^2]^2 + 4(A_x)^2(A_y)^2\cos^2\delta}\}\ , \tag{2b}$$

$$\tan\psi_\pm = \frac{1}{2\cos\delta}\left[\left(\frac{A_y}{A_x} - \frac{A_x}{A_y}\right) \pm \sqrt{\left(\frac{A_y}{A_x} + \frac{A_x}{A_y}\right)^2 - 4\sin^2\delta}\right] , \tag{2c}$$

$$\tan 2\psi = \frac{2A_xA_y\cos\delta}{(A_x)^2 - (A_y)^2} , \tag{2d}$$

where $0 \le I_a < \infty$, $0 \le I_b \le I_a$, and $-\pi/2 \le \psi_\pm \le \pi/2$, with $\psi_\pm$ the angles between the major (+) and minor (−) axes relative to $\hat{\mathbf{x}}$. Although, the expression for $\tan 2\psi$ has two solution $\psi_\pm$, it does not have enough information to know which is which, so we define ψ to be the angle of the major or minor (the smaller in absolute value) axis of the ellipse from $\hat{\mathbf{x}}$, and bound it by $-\pi/4 \le \psi < \pi/4$. In Fig. 1 these variables are shown. To complete this group, we add the V variable, which is one of the real Stokes variables,

$$I = |E_x|^2 + |E_y|^2 = (A_x)^2 + (A_y)^2, \quad 0 < I < \infty \tag{3a}$$

$$Q = |E_x|^2 - |E_y|^2 = (A_x)^2 - (A_y)^2, \quad -\infty < Q < \infty \tag{3b}$$

$$U = E_xE_y^* + E_x^*E_y = 2A_xA_y\cos\delta, \quad -\infty < U < \infty \tag{3c}$$

$$V = i(E_xE_y^* - E_x^*E_y) = 2A_xA_y\sin\delta, \quad -\infty < V < \infty . \tag{3d}$$

The importance of V is its sign, which determines the polarization rotation direction, right or left handed.

III. THE VECTOR STATISTICS

Assuming Gaussian statistics, in the general case, as was shown by Goodman [42] and used by Barakat [28], the basic (field) variables have probabilities which depend upon correlations between all the four components and may be written as

$$P(E_x^r, E_x^i, E_y^r, E_y^i) = \frac{1}{\pi^2\, d}\exp\left(-\frac{1}{d}[\, j_{22}|E_x|^2 + j_{11}|E_y|^2 - 2\,\mathrm{Re}(j_{12}E_x^*E_y)]\right) , \tag{4}$$

where $d = \det\tilde{\mathbf{J}}$, $\tilde{\mathbf{J}}$ is the 2×2 covariant Hermitian matrix ($j_{12} = j_{21}^* = j_{12}^r + ij_{12}^i$),

$$\tilde{\mathbf{J}} = \begin{pmatrix} \langle E_x|E_x\rangle & \langle E_x|E_y\rangle \\ \langle E_y|E_x\rangle & \langle E_y|E_y\rangle \end{pmatrix} , \tag{5}$$

and $\langle E_i|E_j\rangle$ denotes the ensemble average of $E_i^*E_j$. The assumption in Eq. (4) is that $\langle E_x^rE_y^r\rangle = \langle E_x^iE_y^i\rangle = \frac{1}{2}j_{12}^r$ and $\langle E_x^rE_y^i\rangle = -\langle E_x^iE_y^r\rangle = \frac{1}{2}j_{12}^i$. This assumption is justified for independent Feynman paths. Instead of the coherency matrix $\tilde{\mathbf{J}}$, one may use the average Stokes parameters

$$\langle I\rangle = j_{11} + j_{22} , \tag{6a}$$

$$\langle Q\rangle = j_{11} - j_{22} , \tag{6b}$$

$$\langle U\rangle = 2j_{12}^r , \tag{6c}$$

$$\langle V\rangle = 2j_{12}^i , \tag{6d}$$

$$d = \tfrac{1}{4}(\langle I\rangle^2 - \langle Q\rangle^2 - \langle U\rangle^2 - \langle V\rangle^2) , \tag{6e}$$

and degree of polarization P, given by

$$P = \sqrt{1 - \frac{4\det\tilde{\mathbf{J}}}{[\mathrm{tr}(\tilde{\mathbf{J}})]^2}} = \frac{\sqrt{\langle Q\rangle^2 + \langle U\rangle^2 + \langle V\rangle^2}}{\langle I\rangle} . \tag{6f}$$

In this paper, the use of $\tilde{\mathbf{J}}$ is preferred, as the exact form for its matrix elements will be calculated in Sec. IV for various situations. The use of the 4×4 real Mueller matrix, which connects the scattered to incident Stokes parameters, instead of $\tilde{\mathbf{J}}$, is also easily calculated [32] using Eq. (6).

Using Eq. (4) one can find the joint probabilities of the variables in Eqs. (1)–(3). Starting with the amplitude (A) and the phase difference (δ) one gets

$$P(A_x, A_y, \delta) = \frac{1}{\pi^2 d}\int_0^\infty r_1\, r_2\, dr_1\, dr_2 \int_0^{2\pi} d\theta_1\, d\theta_2\, \delta(\delta - [\theta_2 - \theta_1])\, \delta(A_x - r_1) \times \delta(A_y - r_2)\, \exp\left(-\frac{1}{d}\left[j_{22}r_1^2 + j_{11}r_2^2 - 2|j_{12}|r_1r_2\cos(\theta_2 - \theta_1 - \beta)\right]\right) , \tag{7a}$$

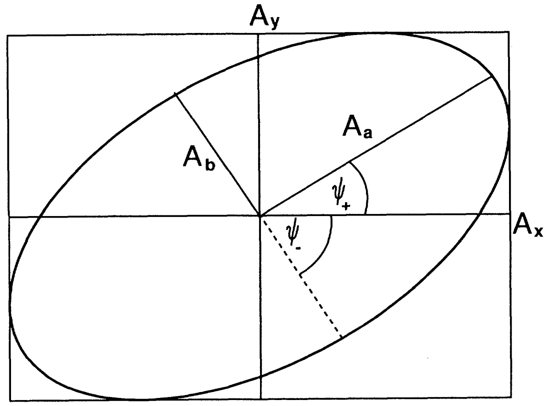

FIG. 1. The ellipse variables A_a and A_b are the amplitudes of the major and minor axes, respectively, while ψ_+ and ψ_- are the angles between $\hat{\mathbf{x}}$ to those axes. A_x and A_y are the amplitudes of the vector field in $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions.

which yields the PDF

$$P(A_x, A_y, \delta) = \frac{(2\pi - |\delta|)}{\pi^2 d} A_x A_y \times \exp\left(-\frac{1}{d}[j_{22}(A_x)^2 + j_{11}(A_y)^2 - 2|j_{12}|A_x A_y \cos(\delta - \beta)]\right), \quad (7b)$$

where $\beta = \arg(j_{12})$. Comparing this with an earlier result by Barakat [28] we note a slight difference due to the factor $(2\pi - |\delta|)/(2\pi)$. However, as Barakat [35] showed, this difference does not affect the probabilities of $\cos\delta$ or $\sin\delta$, which are all that is required here.

From Eq. (7), it is easy to get the intensity statistics of light passed by a polarizer oriented along either x or y. The joint PDF is

$$P(I_x, I_y) = \frac{1}{d}\exp\left(-\frac{1}{d}[j_{22}I_x + j_{11}I_y]\right) \times I_0\left(\frac{2|j_{12}|}{d}\sqrt{I_x I_y}\right) , \quad (8)$$

where $0 \le I_x < \infty$, $0 \le I_y < \infty$, and $I_0()$ is the modified Bessel function of the first kind of order zero [43]. The marginal probabilities of I_x or I_y are the Rayleigh statistics

$$P(I_k) = \frac{1}{\langle I_k\rangle} \exp\left(-\frac{I_k}{\langle I_k\rangle}\right) , \quad (9)$$

where $\langle I_x\rangle = j_{11}$ and $\langle I_y\rangle = j_{22}$. As we will see later, this means that no matter which polarization the incident field has, the marginal probabilities of the scattered field in the x or y direction is independent of the correlation j_{12} and has the form of Eq. (9). The scattered intensity passed by a polarizer in the output beam making an arbitrary angle θ with $\hat{\mathbf{x}}$ ($E_k = E_x\cos\theta + E_y\sin\theta$) gives the same Rayleigh statistics Eq. (9), with $\langle I_k\rangle = j_{11}\cos^2\theta + j_{22}\sin^2\theta + |j_{12}|\cos\beta\sin 2\theta$.

Defining the correlation coefficient between two variables g, h to the powers m, n as

$$C_{mn}(g,h) = \frac{\langle g^m h^n\rangle}{\langle g^m\rangle\langle h^n\rangle} , \quad (10)$$

then

$$C_{mn}(I_x, I_y) = \left(\frac{d}{j_{11}j_{22}}\right)^n {}_2F_1\left(-n, m+1; 1; -\frac{|j_{12}|^2}{d}\right) , \quad (11)$$

where ${}_2F_1(a,b;c;z)$ is the degenerate hypergeometric function [43].

The most interesting variables are those of the ellipse. Barakat [28] gives the joint and marginal probabilities of I, ψ_+, and ϵ_B, where ϵ_B is Barakat's ellipticity [see the discussion after Eq. (21)]. Here we give results for additional ellipse variables which relate to recent experiments on the vector statistics [29,37,41]. The calculation given here for correlated fields uses the method employed before by Cohen *et al.* [29] for independent fields.

The joint probability of I_a, I_b, and $\psi_\pm$ is given by

$$P_\pm(I_a, I_b, \psi_\pm) = \frac{1}{\pi d}\frac{I_a - I_b}{\sqrt{I_a I_b}} \times \exp\left(-\frac{(j_{11}+j_{22})(I_a+I_b)}{2d}\right) \times \cosh\left(\frac{2|j_{12}|}{d}\sin\beta\sqrt{I_a I_b}\right) \times \cosh[Z(\psi_\pm)(I_a - I_b)] , \quad (12)$$

where $0 \le I_b \le I_a < \infty$ and $Z(\psi_\pm) = \frac{|j_{12}|}{d}\cos\beta\sin 2\psi_\pm + \frac{(j_{11}-j_{22})}{2d}\cos 2\psi_\pm$. From Eq. (12) we see that $P_\pm(I_a, I_b = I_a, \psi_\pm) = 0$, which means that the probability to have circular polarization is zero for any system and for any input polarization (except when $d \equiv 0$, which corresponds, for example, to circular input polarization and unit degree of polarization; see Sec. IV). The term

$$\left[\cosh\left(\frac{2|j_{12}|}{d}\sin\beta\sqrt{I_a I_b}\right)\right]$$

can be written as the sum of the two terms

$$\left[\exp\left(\pm\frac{2|j_{12}|}{d}\sin\beta\sqrt{I_a I_b}\right)\right],$$

which are related to the difference in the contribution of the right- or left-handed polarization rotation direction to the probability.

Integrating on $\psi_\pm$ gives

$$P(I_a,I_b) = \frac{I_a - I_b}{d\sqrt{I_aI_b}} \exp\left(-\frac{1}{2d}(j_{11}+j_{22})(I_a+I_b)\right) \cosh\left(\frac{2|j_{12}|}{d}\sin\beta\,\sqrt{I_aI_b}\right) \times I_0\left(\frac{(I_a-I_b)}{2d}\sqrt{4|j_{12}|^2\cos^2\beta + (j_{11}-j_{22})^2}\right) . \tag{13}$$

It is important to note the difference between I_x, I_y to I_a, I_b. The first pair is measured in a fixed direction in space while the last pair has axes which are not fixed in space (or time for temporal randomness) but change for each speckle spot (or instant).

The intensity in a single speckle spot as a function of the direction of rotation polarizer making an angle φ to the x axes located in the output beam can be written as $I(\varphi) = a\cos^2(\varphi - \psi_+) + b$. The joint probabilities of a, b, and ψ_+ is easy to determine. Using the relation $a = I_a - I_b$, $b = I_b$ we get

$$P_{a,b,\psi_\pm}(a,b,\psi_\pm) = \begin{cases} P_\pm(I_a = a+b, I_b = b, \psi_\pm), \\ \qquad 0 \le a, b < \infty \\ 0 \ \text{otherwise.} \end{cases} \tag{14}$$

The marginal probabilities of $I = I_a + I_b$ and $a = I_a - I_b$ can be calculated more easily than those of I_a and I_b (the marginal probabilities of I_a and I_b are given in Sec. IV for spacial forms of the covariance matrix). The PDF of I for $P \neq 0$ is given by

$$P(I) = \exp\left(-\frac{(j_{11}+j_{22})}{2d} I\right) \times \frac{\sinh\left(\frac{I}{d}\sqrt{|j_{12}|^2 + \frac{1}{4}(j_{11}-j_{22})^2}\right)}{\sqrt{|j_{12}|^2 + \frac{1}{4}(j_{11}-j_{22})^2}} \tag{15a}$$

or, in more familiar form,

$$P(I) = \frac{1}{P\langle I\rangle}\left[\exp\left(-\frac{2I}{(1+P)\langle I\rangle}\right) - \exp\left(-\frac{2I}{(1-P)\langle I\rangle}\right)\right] , \tag{15b}$$

while for $P = 0$

$$P(I) = \frac{4I}{\langle I\rangle^2} \exp\left(-\frac{2I}{\langle I\rangle}\right) , \tag{15c}$$

a result which was obtained previously [28, 44]. On the other hand,

$$P(a) = \frac{a}{d} I_0\left(\frac{a}{2d}\sqrt{(j_{11}-j_{22})^2 + 4|j_{12}|^2\cos^2\beta}\right) \times K_0\left(\frac{a}{2d}\sqrt{(j_{11}+j_{22})^2 - 4|j_{12}|^2\sin^2\beta}\right), \tag{16}$$

where $K_0(\,)$ is the modified Bessel function of the second kind of order zero [43].

Instead of an unsolvable integral equation for the moments of the major (I_a) and minor (I_b) intensities, one can use the connection to the correlation between I and a ($I_a = \frac{1}{2}[I+a]$ and $I_b = \frac{1}{2}[I-a]$)

$$\langle I_a^n\rangle = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\langle I^{n-k}a^k\rangle ,$$
$$\langle I_b^n\rangle = \frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\langle I^{n-k}a^k\rangle , \tag{17}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and

$$\langle I^m a^n\rangle = \left(-\frac{2d}{j_{11}+j_{22}}\right)^m \left[\frac{\partial^m T_n(\nu)}{\partial\nu^m}\right]_{\nu=1} , \tag{18a}$$

$$T_n(\nu) = \frac{(4d)^{n+1}\left[\Gamma(n/2+1)\right]^2}{\left[\nu^2(j_{11}+j_{22})^2 - 4(j_{12}^i)^2\right]^{n/2+1}} {}_2F_1\left(\frac{n}{2}+1, \frac{n}{2}+1; 1; \frac{(j_{11}-j_{22})^2 + 4(j_{12}^r)^2}{\nu^2(j_{11}+j_{22})^2 - 4(j_{12}^i)^2}\right) , \tag{18b}$$

where $\Gamma(\,)$ is the gamma function[43]. The averaged major and minor intensities are

$$\langle I_a\rangle = \frac{1}{2}[\langle I\rangle + T_1(1)] , \tag{19a}$$

$$\langle I_b\rangle = \frac{1}{2}[\langle I\rangle - T_1(1)] . \tag{19b}$$

For unpolarized light ($P = 0$) $T_1(1) = \frac{\pi}{4}\langle I\rangle$.

The ellipticity of the ellipse is defined as the ratio of the minor to major axes ($\epsilon = A_b/A_a$, $0 \le \epsilon \le 1$). Its joint probability, with $\psi_\pm$, is given by

$$P_\pm(\epsilon,\psi_\pm) = \frac{(1-\epsilon^2)}{\pi d}\left[\frac{1}{(A+B\mp C)^2}+\frac{1}{(A-B\mp C)^2}\right] \tag{20}$$

where $A = \frac{1}{2d}(j_{11}+j_{22})(1+\epsilon^2)$, $B = \frac{2}{d}|j_{12}|\epsilon\sin\beta$, and $C = Z(\psi_\pm)(1-\epsilon^2)$. Note that $P_\pm(\epsilon=1,\psi_\pm)$ does not depend on $\psi_\pm$ as required from symmetry for circular polarization, which has equal major and minor axes in any arbitrary direction. Also $P_\pm(\epsilon=1,\psi_\pm)=0$ unless $d\equiv 0$. ϵ and $\psi_\pm$ are independent when $\psi_\pm$ is uniformly distributed ($\beta=\pi/2$ or $3\pi/2$ and $j_{11}=j_{22}$).

Integrating on $\psi_\pm$ gives

$$P(\epsilon) = d(1-\epsilon^2)\left[\frac{\omega_+}{(\omega_+^2-\eta^2)^{3/2}}+\frac{\omega_-}{(\omega_-^2-\eta^2)^{3/2}}\right] , \tag{21a}$$

where

$$\eta = (1-\epsilon^2)\sqrt{|j_{12}|^2\cos^2\beta+\frac{(j_{22}-j_{11})^2}{4}} , \tag{21b}$$

$$\omega_\pm = \frac{1}{2}(j_{11}+j_{22})(1+\epsilon^2)\pm 2\epsilon|j_{12}|\sin\beta . \tag{21c}$$

Our definition for the ellipticity (ϵ) differs from that of Barakat (ϵ_B) [28] where $-1\le\epsilon_B\le 1$, as $\epsilon_B = \epsilon\ \mathrm{sgn}(\sin\delta)$, where $\mathrm{sgn}(\sin\delta)$ determines the polarization rotation direction [this is related to the two terms in each of Eqs. (20) and (21a), see the discussion after Eq. (12)]. The probabilities have the relation $P(\epsilon) = P_{\epsilon_B}(\epsilon)+P_{\epsilon_B}(-\epsilon)$ [which is equal to the contributions of the two terms in Eq. (21a)], where $P_{\epsilon_B}(\epsilon_B)$ was also given before in Eq. (5.9) in Ref. [28]—the last term in that equation should be

$$P^2\left[1+\left(\frac{2\epsilon_1}{1+\epsilon_1^2}\right)^2\left(\frac{2\epsilon}{1+\epsilon^2}\right)^2\right]$$

instead of

$$2P\left(\frac{2\epsilon_1}{1+\epsilon_1^2}\right)\left(\frac{2\epsilon}{1+\epsilon^2}\right).$$

On the other hand, integrating Eq. (20) on ϵ gives the marginal probability of $\psi_\pm$

$$P_\pm(\psi_\pm) = \frac{4}{\pi d\Omega}\left\{1+\frac{(G_\pm-G_\mp)}{\sqrt{\Omega}}\times\left[\arctan\left(\frac{H+2G_\pm}{\sqrt{\Omega}}\right)-\arctan\left(\frac{H-2G_\pm}{\sqrt{\Omega}}\right)\right]\right\} , \tag{22}$$

where $G_+(\psi_\pm) = \frac{(j_{11}+j_{22})}{2\,d}+Z(\psi_\pm)$, $G_-(\psi_\pm) = \frac{(j_{11}+j_{22})}{2\,d}-Z(\psi_\pm)$, $H=\frac{2}{d}|j_{12}|\sin\beta$, and $\Omega = 4G_+\,G_- - H^2$.

This last result coincides with Barakat's [28] result for $P_+(\psi_+)$. Here we point out that the $\tan 2\psi$ expression [Eq. (2)] can give [29] the probabilities which include $\psi_\pm$ by using the fact that $P(\psi) = P_+(\psi)+P_-(\psi)$ and $P_+(\psi) = P_-(\psi+\pi/2)$. Note that the term responsible for breaking the symmetry of $P_\pm(\psi_\pm)$ around $\psi_\pm=0$ is just j_{12}^r. This is related to the possibility of choosing different coordinate system $(\hat{\mathbf{x}}',\hat{\mathbf{y}}')$ instead of $\hat{\mathbf{x}},\hat{\mathbf{y}}$ by simple rotation in the x-y plane, to get a transformed covariance matrix $\tilde{\mathbf{J}}'$ with $j_{1'2'}^r=0$.

Fercher and Steeger [30, 39] initially calculated the statistics of the Stokes variables for independent fields (A_x, A_y, and δ independent), and later [40] they recalculated these statistics assuming correlations between A_x, A_y but with δ independent (uniformly distributed). Barakat [35] gives these statistics assuming that $j_{12}^i = 0$ ($\beta=0$), meaning $\langle V\rangle = 0$. Here we give the marginal probabilities of these variables for *fully* correlated fields [using Eq. (4)]. The intensity PDF was given in Eq. (15) while the PDF of Q is easily derived from Eq. (8),

$$P(Q) = \frac{\exp\left(\frac{1}{2d}\left[Q(j_{11}-j_{22})-|Q|\sqrt{(j_{11}+j_{22})^2-4|j_{12}|^2}\right]\right)}{\sqrt{(j_{11}+j_{22})^2-4|j_{12}|^2}} . \tag{23}$$

It is easier to get the marginal probabilities of U and V starting from Eq. (4),

$$P(U) = \frac{\exp\left(-\frac{1}{d}\left[-U|j_{12}|\cos\beta+|U|\sqrt{j_{11}j_{22}-|j_{12}|^2\sin^2\beta}\right]\right)}{2\sqrt{j_{11}j_{22}-|j_{12}|^2\sin^2\beta}} \tag{24}$$

and

$$P(V) = \frac{\exp\left(-\frac{1}{d}(-V|j_{12}|\sin\beta+|V|\sqrt{j_{11}j_{22}-|j_{12}|^2\cos^2\beta})\right)}{2\sqrt{j_{11}j_{22}-|j_{12}|^2\cos^2\beta}} . \tag{25}$$

These marginal probabilities coincide with Barakat [35] results for $\beta=0$. Since in general $\beta\neq 0$, the Goodman

[42] type of multivariable complex Gaussian probability [Eq. (4)] is now seen to have the significance (unlike previous claims [36]) needed in optics.

The statistics of the last three Stokes variables [Eqs. (23)–(25)] can be simply written as the universal function

$$P(y) = \frac{\exp\left(\frac{1}{2d}\left[y\langle y\rangle - |y|\sqrt{\langle I\rangle^2(1-P^2)+\langle y\rangle^2}\right]\right)}{\sqrt{\langle I\rangle^2(1-P^2)+\langle y\rangle^2}} , \tag{26a}$$

with moments

$$\langle y^n\rangle = \frac{n!}{2^{n+1}\sqrt{\langle I\rangle^2(1-P^2)+\langle y\rangle^2}}\left\{[\sqrt{\langle I\rangle^2(1-P^2)+\langle y\rangle^2}+\langle y\rangle]^{n+1} + (-1)^n[\sqrt{\langle I\rangle^2(1-P^2)+\langle y\rangle^2}-\langle y\rangle]^{n+1}\right\}, \tag{26b}$$

where $y = Q$, U, or V. These moments coincide with the first three cumulants which were given in Ref. [36] for y.

The four Stokes variables are connected by $I^2 = Q^2 + U^2 + V^2$ for each point in time and space. The reason is because of the infinite possibilities of the fields with the same relative phase δ but with different δ_x [Eq. (3)] that give the same Stokes variables which, like the ellipse variables, do not depend on δ_x.

As mentioned, the direction, right or left handed, of the field polarization state is determined by the sign of V. The parts of the ellipse in the scattered speckle spots which have right- or left-handed ($\pm$) polarization direction ($0 \le \mu_\pm \le 1$) is determined by integrating $P(V)$ from 0 to ∞ (μ_+) or from $-\infty$ to 0 (μ_-),

$$\mu_\pm = \frac{d}{2\sqrt{d+(j_{12}^i)^2}\,[\sqrt{d+(j_{12}^i)^2} \mp j_{12}^i]} , \tag{27a}$$

or, in form of the Stokes parameters,

$$\mu_\pm = \left[2\sqrt{1+\frac{\langle V\rangle^2}{\langle I\rangle^2(1-P^2)}} \times\left(\sqrt{1+\frac{\langle V\rangle^2}{\langle I\rangle^2(1-P^2)}} \mp \frac{\langle V\rangle}{\langle I\rangle\sqrt{(1-P^2)}}\right)\right]^{-1} . \tag{27b}$$

IV. THE COVARIANCE MATRIX

Until now our calculations were general and did not assume anything about the source of the randomness (temporal or speckle patterns) beside the fact that it is Gaussian correlated. The forms of the Jones and Mueller matrices were investigated before [45], assuming that the spatial fluctuations vary more slowly than those of the temporal fluctuations caused mainly by the input wave. A simple theory is now given for stationary spatial randomness (speckle patterns). The dependence of the covariant Hermitian coherency matrix $\tilde{\mathbf{J}}$ (for multiple scattering) on the incident polarization state and the scattering medium with slab geometry for normally incident and scattered wave vectors is derived using the method of Ref. [41] (for the general case of arbitrary incident and scattered wave vector see Ref. [32]).

The scattered field ($\mathbf{E}_s$), on the output surface, is connected to the incident field ($\mathbf{E}_i$), on the input surface, using the scattering matrix $\tilde{\mathbf{F}}$ (where F_{ij} is the response of the medium to a unit amplitude input in the i direction which produces a scattered field in the j direction $i,j = x,y,z$). In this fashion the coherency matrix $\tilde{\mathbf{J}}$ in the general case can be written as

$$\tilde{\mathbf{J}} = \langle \mathbf{E}_s\, \mathbf{E}_s^\dagger\rangle = \langle \tilde{\mathbf{F}}^T\, \tilde{\mathbf{J}}_i\, (\tilde{\mathbf{F}}^T)^\dagger\rangle , \tag{28}$$

where $\mathbf{E}_s = \tilde{\mathbf{F}}^T\mathbf{E}_i$ and $\tilde{\mathbf{J}}_i = \mathbf{E}_i\mathbf{E}_i^\dagger$ is the covariance matrix of the incident field.

We see that $\tilde{\mathbf{J}}$ includes correlations between the elements of $\tilde{\mathbf{F}}$. These correlations can be determined from the symmetry of the system. In general there are 4 output fields components (in the xy plane) and thus 16 potentially different correlators, but if the random medium is statistically isotropic, then the average rotational symmetry requires that all correlators be real and that only those correlators in which a given direction (x or y) appears an even number of times can be nonzero. In addition, time-reversal symmetry requires $\langle|F_{ij}|^2\rangle = \langle|F_{ji}|^2\rangle$ for all i,j [46]. It is convenient to normalize all the correlators to the same scale dividing by $\langle|F_{xx}|^2\rangle = \langle|F_{yy}|^2\rangle$, which is connected to the average scattered intensity $\langle I\rangle = \langle|E_x|^2\rangle + \langle|E_y|^2\rangle$. Since the scattered field is built of many Feynman paths, each term of the $\tilde{\mathbf{F}}$ matrix can be written [47, 48]

$$F_{ij} = \sum_{n=0}^{\infty} a_{ij}(n)\exp(i\Phi_n) . \tag{29}$$

If [44] the amplitudes $a_{ij}(n)$ and the phase $\Phi_{n'}$ are statistically independent of each other, and the phases are uniformly distributed on the primary interval $(-\pi,\pi)$, then the only correlations are between the real parts of F_{ij} or between the imaginary parts of F_{ij}. Further, it is easily seen that $\langle|F_{ij}^r|^2\rangle = \langle|F_{ij}^i|^2\rangle$, which satisfies the assumption used in Eq. (4). This leaves three material parameters [32, 38, 41, 47, 48], which following previous notation we label

$$\rho = \langle|F_{xy}|^2\rangle , \tag{30a}$$

$$\Gamma = \langle F_{xx}|F_{yy}\rangle , \tag{30b}$$

$$\Delta = \langle F_{xy} | F_{yx} \rangle \ . \tag{30c}$$

There is an exact sum rule which is a result of symmetry [38, 41]

$$1 - \rho = \Gamma + \Delta \ . \tag{31}$$

The effect of an additional optical element on the covariance matrix and especially on the statistics may be easily determined by multiplying $\tilde{\mathbf{F}}^T$, to the right (left), with the Jones matrix which describes these elements before (after) the scattering medium. The covariance matrix $\tilde{\mathbf{J}}$ can be diagonalized by a matrix transformation, which is the Jones matrix [49] representation of a coordinate rotation, a relative retardation of the two components or a combination of both [32,50–54].

A few examples are given to illustrate the use of this method. First, for linearly polarized light in the $\hat{\mathbf{x}}$ direction which is normally incident and normally scattered to the slab surfaces,

$$\tilde{\mathbf{J}} = \frac{\langle I \rangle}{1+\rho} \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}, \tag{32}$$

where $\langle |F_{xx}|^2 \rangle (1+\rho) = \langle I \rangle$ and $P = \frac{1-\rho}{1+\rho}$. Using Eq. (32) all the results of Ref. [29] are obtained. The statistics of I_a and I_b, for arbitrary ρ, is given by

$$P(I_a) = 2k_2 I_a \ \exp(-k_1 I_a) \sum_{n=0}^{\infty} \frac{(2n)!(k_3 I_a)^{2n}}{(n!)^2 (4n+1)!!} \times \left[{}_1F_1\left(\frac{1}{2}; 2n+\frac{3}{2}; -k_1 I_a\right) - \frac{1}{4n+3} \, {}_1F_1\left(\frac{3}{2}; 2n+\frac{5}{2}; -k_1 I_a\right) \right] , \tag{33a}$$

where ${}_1F_1(a;b;c)$ is the degenerate hypergeometric function [43], and

$$P(I_b) = k_2 I_b \exp\left(-\frac{3k_1 I_b}{2}\right) \sum_{n=0}^{\infty} \frac{(2n)!(k_3 I_b)^{2n}}{(n!)^2 2^{2n}} (k_1 I_b)^{-(n+3/4)} \times \left[(k_1 I_b)^{-1/2} W_{-n+1/4, n+3/4}(k_1 I_b) - W_{-n-1/4, n+1/4}(k_1 I_b) \right] , \tag{33b}$$

where $W_{\alpha,\beta}(z)$ is the Whittaker function [43] and

$$k_1 = \frac{(1+\rho)^2}{2\rho \langle I \rangle} \ , \quad k_2 = \frac{(1+\rho)^2}{\rho \langle I \rangle^2} \ , \quad k_3 = \frac{(1-\rho^2)}{2\rho \langle I \rangle} \ . \tag{33c}$$

A typical joint probability $P(\epsilon, \psi_+)$ [Eq. (20)] is plotted in Fig. 2 for $\rho = 1/2$. The most probable value is for linear polarization ($\epsilon = 0$) in the x direction ($\psi_+ = 0$), corresponding to a partial memory of the incident polarization direction.

Suppose that the incident polarization is elliptical with major and minor axes in the $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ direction [$\mathbf{E}_i = (\sqrt{\sigma}, i)$] and the incident and scattered wave vector is normal to the surface of the slab ($\hat{\mathbf{z}}$). Then the scattered covariance matrix is

$$\tilde{\mathbf{J}} = \langle I \rangle \frac{1+\sigma\rho}{(1+\sigma)(1+\rho)} \times \begin{pmatrix} \dfrac{\sigma+\rho}{1+\sigma\rho} & \dfrac{i\sqrt{\sigma}(\Gamma-\Delta)}{\sqrt{(\sigma+\rho)(1+\sigma\rho)}} \\ \dfrac{-i\sqrt{\sigma}(\Gamma-\Delta)}{\sqrt{(\sigma+\rho)(1+\sigma\rho)}} & 1 \end{pmatrix}, \tag{34}$$

so $j_{12}^r = 0$ and

$$P = \sqrt{1 + 4\frac{\sigma(\Gamma-\Delta)^2 - (\sigma+\rho)(1+\sigma\rho)}{(1+\rho)^2(1+\sigma)^2}} .$$

Figure 3 shows a typical joint distribution function of ϵ, ψ_+ [Eq. (20)] for $\sigma > 1$ (major input polarization axes in the x direction) while Fig. 4 give these statistics for $\sigma < 1$ (the input major axes in the y direction). Again the most probable value ($\epsilon > 0$ while $\psi_+ = 0$ in Fig. 3 and $\psi_+ = \pm\pi/2$ in Fig. 4) shows that the input polar-

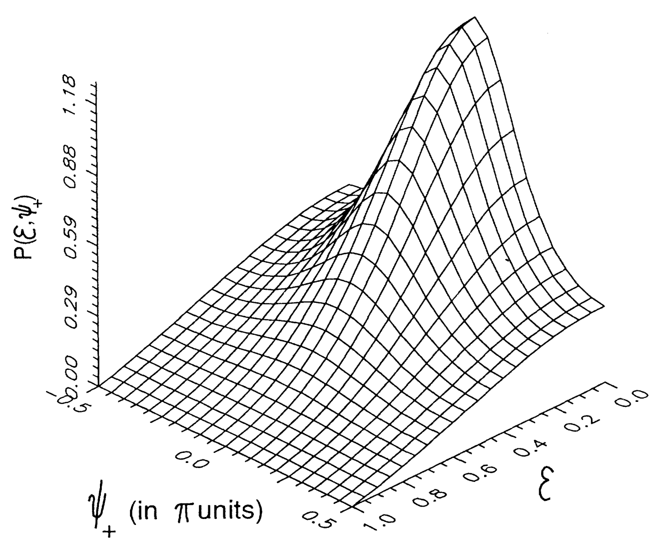

FIG. 2. The joint distribution of ϵ, ψ_+ [Eq. (20)] for the explicit covariance matrix in Eq. (32), where $\rho = 1/2$. The most probable output polarization is for $\epsilon = 0$ (linear polarization state) in the x direction ($\psi_+ = 0$).

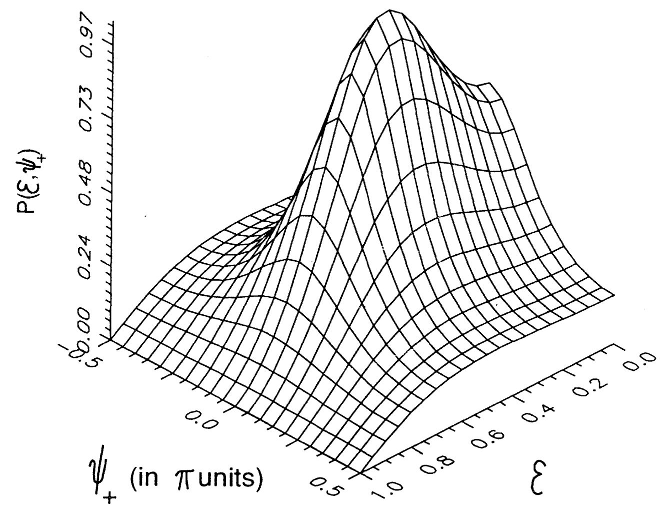

FIG. 3. The joint distribution of ϵ, ψ_+ [Eq. (20)] for elliptic polarization in the input with major axes in the x direction ($\sigma < 1$) for the explicit covariance matrix: $j_{11} = 1$, $j_{22} = 1/2$, $j^i_{12} = 1/2$, and $j^r_{12} = 0$.

ization state is "remembered" in the output beam.

For circular polarization ($\sigma = 1$) in the input,

$$\tilde{\mathbf{J}} = \frac{\langle I\rangle}{2}\begin{pmatrix} 1 & iP \\ -iP & 1 \end{pmatrix}, \tag{35}$$

where $P = (\Gamma - \Delta)/(1+\rho)$ is the degree of polarization. The marginal probabilities of the major and minor axes in the output beam are given by

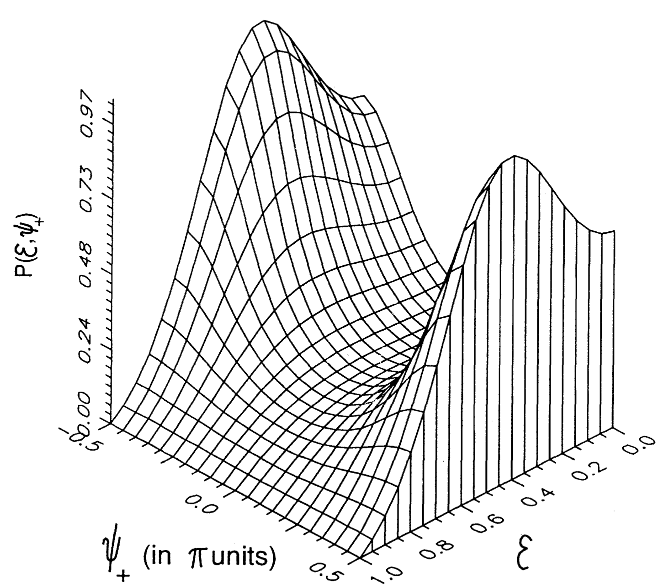

FIG. 4. The joint distribution of ϵ, ψ_+ [Eq. (20)] for elliptic polarization in the input with major axes in the y direction ($\sigma > 1$) for the explicit covariance matrix: $j_{11} = 1/2$, $j_{22} = 1$, $j^i_{12} = 1/2$, and $j^r_{12} = 0$.

$$P(I_a) = F_2(I_a) + F_3(I_a)\,, \tag{36a}$$

$$P(I_b) = F_2(I_b) + F_3(I_b) - F_1(I_b)\,, \tag{36b}$$

$$F_1(x) = \frac{2\sqrt{\pi}}{\langle I\rangle}\exp\left(\frac{-2x}{\langle I\rangle}\right)\left[\sqrt{\frac{2x(1-P^2)}{\langle I\rangle}} - \sqrt{\frac{\langle I\rangle(1-P^2)}{8x}}\right], \tag{36c}$$

$$F_2(x) = \frac{1}{2}F_1(x)\left[\mathrm{erf}\left(\sqrt{\frac{2x(1-P)}{\langle I\rangle(1+P)}}\right) + \mathrm{erf}\left(\sqrt{\frac{2x(1+P)}{\langle I\rangle(1-P)}}\right)\right], \tag{36d}$$

$$F_3(x) = \frac{2}{\langle I\rangle}\exp\left(-\frac{4x}{\langle I\rangle(1-P^2)}\right) \times\left[P\sinh\left(\frac{4xP}{\langle I\rangle(1-P^2)}\right) + \cosh\left(\frac{4xP}{\langle I\rangle(1-P^2)}\right)\right], \tag{36e}$$

where erf( ) is the error function [43]. For $P = 1$, $F_1(x) = 0$, meaning that I_a and I_b have the same PDF form while for $P \neq 1$, $P(I_a = 0) = 0$ and $P(I_b \to 0) \sim (I_b)^{-1/2}$. Using Eq. (36), the marginal probabilities of the major (A_a) and minor (A_b) amplitudes are shown in Fig. 5 for different values of degree of polarization. Also $P_\pm(\psi_\pm) = 1/\pi$ as required from symmetry.

The correlation $C_{mn}(I_a, I_b)$ is related to

$$\langle I_a^m I_b^n\rangle = \langle I\rangle^{m+n}\frac{(1-P^2)^{m+n+1}}{2^{2m+2n+1}} \times\sum_{k=0}^{\infty}\frac{P^{2k}}{(2k)!}(m+n+2k+1)! \times\left[\frac{R_{mnk}(n+k+3/2)}{(2n+2k+1)} - \frac{R_{mnk}(n+k+5/2)}{(2n+2k+3)}\right], \tag{36f}$$

where $R_{mnk}(x) = {}_2F_1(1, m+n+2k+2; x; 1/2)$. The PDF of ϵ is

$$P(\epsilon) = (1-P^2)(1-\epsilon^2)\left[\frac{1}{(1+\epsilon^2+2P\epsilon)^2} + \frac{1}{(1+\epsilon^2-2P\epsilon)^2}\right]. \tag{37}$$

$P(\epsilon)$ has a maximum for $\epsilon = \epsilon_{\max}$, which is a function of P. For $P < 1/2$, $\epsilon_{\max} = 0$, meaning that it is more probable to have linear polarization than any other polarization. For $P > 1/2$, $\epsilon_{\max}$ is finite, and goes to 1 when there is no scattering at all, in which case $P = 1$.

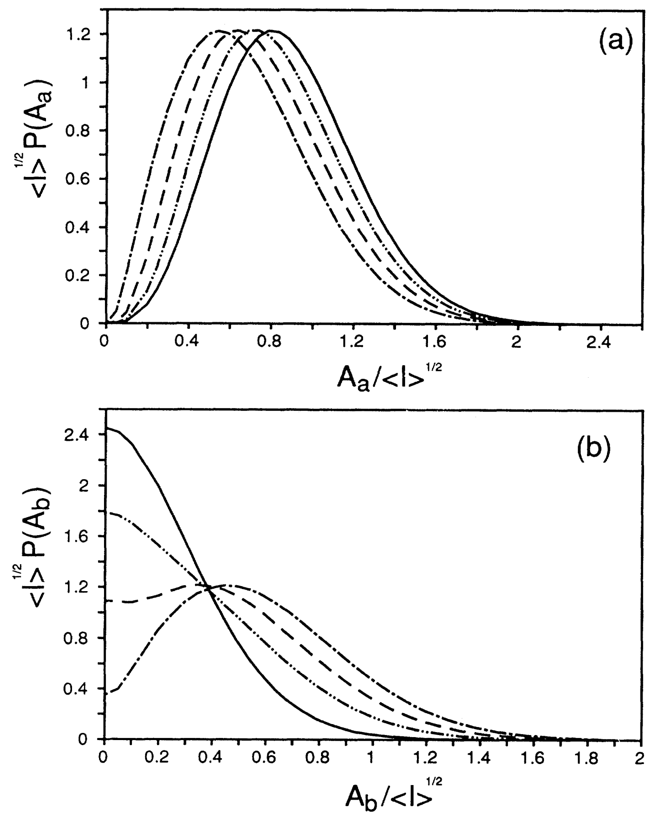

FIG. 5. The statistics of the (a) major and (b) minor amplitudes is plotted using Eq. (36) for different degrees of polarization values. $P = 0.2$ for the solid line, while $P = 0.7$ for the dashed–double-dotted line, $P = 0.9$ for the dashed line, and the dash-dotted line shows the statistics for $P = 0.99$.

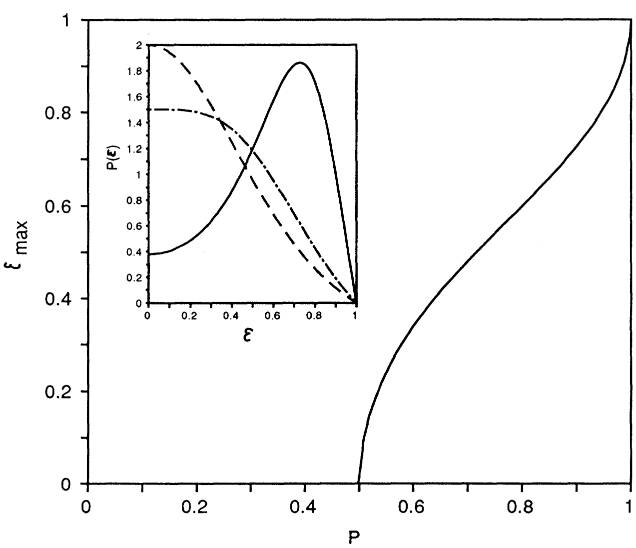

FIG. 6. ϵ_{max}, which characterize the maximum of $P(\epsilon)$, vs the degree of polarization P for the covariance matrix in the form of Eq. (35) ($j_{11} = j_{22}$ and $j_{12}^r = 0$). The inset show the statistics of the ellipticity [Eq. (37)], for the same covariance matrix. For the dash-line $P = 0$, while $P = 0.5$ for the dash-dotted line and $P = 0.9$ for the solid line.

ϵ_{max} as a function of P is shown in Fig. 6, while in the inset, $P(\epsilon)$ versus ϵ [Eq. (37)], is plotted, for different values of P. $\mu_\pm = \frac{1}{2}(1 \pm P)$, meaning that at least half of the speckles spots "remember" the incident polarization direction and that it goes linearly with the degree of polarization.

The statistics of circular input polarization for $P = 0$ is the same as for $\rho = 1$ when the input light is linearly polarized, where in the general case this is the statistics when the scattered light is unpolarized and does not depend on the incident polarization.

Circular polarization state in the input could be achieved with linear polarization followed by a quarter-wave plate whose fast axis is in 45° degree to the incident wave polarization direction. An interesting question is what will happen if the quarter-wave plate is moved after the scattering media. The calculation give the same covariance matrix as Eq. (35) but with $P = \frac{(1-\rho)}{(1+\rho)}$.

Rotating the input direction of a linearly input polarization, with normally incident and scattered wavevector, $\mathbf{E}_i = (\cos\varphi, \sin\varphi)$, with no polarizer in the output, the output intensity for any speckle spot (not just for the averaged) is

$$I(\varphi) = a\cos^2(\varphi-\alpha)+b = A\cos^2\varphi+\frac{1}{2}B\sin 2\varphi+C\sin^2\varphi \;, \qquad (38)$$

where the first form was given in [41] and where measurements for $\rho = 1$ there performed. More experimental results [37] were given later for arbitrary ρ with the suggestion that ρ is not the only parameter needed to describe the statistics. In Eq. (38) $0 \le a,\, b < \infty$, $0 \le \alpha < \pi$, $0 \le A, C < \infty$, $-\infty < B < \infty$, and

$$a = \sqrt{(A-C)^2 + B^2} \;, \qquad (39a)$$

$$b = \tfrac{1}{2}\left[A + C - \sqrt{(A-C)^2 + B^2}\right] \;, \qquad (39b)$$

$$\alpha = \tfrac{1}{2}\arctan\left(\frac{A-C}{B}\right) \;; \qquad (39c)$$

also

$$A = |F_{xx}|^2 + |F_{yx}|^2 \;, \qquad (40a)$$

$$B = 2\,\mathrm{Re}(F_{xx}F_{yx}^* + F_{xy}F_{yy}^*) \;, \qquad (40b)$$

$$C = |F_{xy}|^2 + |F_{yy}|^2 \;. \qquad (40c)$$

The axes $\hat{\varphi}$ ($\varphi = 0$) is chosen arbitrarily, which teaches a few points about the statistics. First we should understand that a, b are coordinate (φ) independent while α, A, B, and C change with the choice of coordinate system. From symmetry this means that

$$P(\alpha) = \frac{1}{\pi} \quad , \tag{41a}$$

$$P_{A',B',C'}(A',B',C') = P_{A,B,C}(A',B',C') = P_{A,B,C}(A(A',B',C'),B(A',B',C'),C(A',B',C')) \quad , \tag{41b}$$

where

$$A(A',B',C') = A'\cos^2\vartheta - \tfrac{1}{2}B'\sin 2\vartheta + C'\sin^2\vartheta \quad , \tag{41c}$$

$$B(A',B',C') = \tfrac{1}{2}(A'\sin 2\vartheta + B'\cos 2\vartheta - C'\sin 2\vartheta) \quad , \tag{41d}$$

$$C(A',B',C') = A'\sin^2\vartheta - \tfrac{1}{2}B'\sin 2\vartheta + C'\cos^2\vartheta, \tag{41e}$$

for arbitrary ϑ, and where A, B, C are measured for the coordinate $\hat{\varphi}$, and A', B', C' are measured for the coordinate $\hat{\varphi}'$ (the angle between $\hat{\varphi}$ to $\hat{\varphi}'$ is ϑ). In Eq. (41b) the use of the unity Jacobian of the transformation was taken into account $\left(\frac{\partial(A,B,C)}{\partial(A',B',C')} = 1\right)$.

Assuming Gaussian fields we have

$$P(F^r_{xx},F^i_{xx},F^r_{xy},F^i_{xy},F^r_{yx},F^i_{yx},F^r_{yy},F^i_{yy}) = P_1(F^r_{xx},F^i_{xx},F^r_{yy},F^i_{yy})\; P_2(F^r_{xy},F^i_{xy},F^r_{yx},F^i_{yx}) \quad , \tag{42a}$$

$$P_1(F^r_{xx},F^i_{xx},F^r_{yy},F^i_{yy}) = \frac{(1+\rho)}{\langle I\rangle}\frac{1}{\pi^2 d_1}\exp\left(-\frac{1}{d_1}[|F_{xx}|^2+|F_{yy}|^2+2\Gamma(F^r_{xx}F^r_{yy}+F^i_{xx}F^i_{yy})]\right) \quad , \tag{42b}$$

$$P_2(F^r_{xy},F^i_{xy},F^r_{yx},F^i_{yx}) = \frac{(1+\rho)}{\langle I\rangle}\frac{1}{\pi^2 d_2}\exp\left(-\frac{1}{d_2}[\rho(|F_{xy}|^2+|F_{yx}|^2)+2\Delta(F^r_{xy}F^r_{yx}+F^i_{xy}F^i_{yx})]\right) \quad , \tag{42c}$$

where $d_1 = \langle I\rangle\frac{1-\Gamma^2}{1+\rho}$ and $d_2 = \langle I\rangle\frac{\rho^2-\Delta^2}{1+\rho}$.

As a first result for the statistics of these A, B, C variables, the joint probability of A and C is

$$\begin{aligned} P(A,C) = {} & \frac{(1+\rho)^2}{d_1 d_2\langle I\rangle^2}\exp\left(-\frac{1}{d_1}[A+C]\right) \\ & \times\sum_{m,n=0}^{\infty}\left(\frac{\Gamma}{d_1}\right)^{2m}\left(\frac{\Delta}{d_2}\right)^{2n}\frac{(AC)^{m+n+1}}{[(m+n+1)!]^2} \\ & \times{}_1F_1\left(n+1;m+n+2;\left(\frac{1}{d_1}-\frac{\rho}{d_2}\right)A\right){}_1F_1\left(n+1;m+n+2;\left(\frac{1}{d_1}-\frac{\rho}{d_2}\right)C\right). \end{aligned} \tag{43}$$

The marginal probabilities of both A and C have the same functional form

$$P(A) = \frac{(1+\rho)^2}{\rho\langle I\rangle^2}\exp\left(-\frac{A}{d_1}\right)\sum_{m,n=0}^{\infty}\frac{\Gamma^{2m}}{d_1^m}\frac{\Delta^{2n}}{d_2^n}\frac{1}{\rho^n}\frac{A^{m+n+1}}{(m+n+1)!}{}_1F_1\left(n+1;m+n+2;\left(\frac{1}{d_1}-\frac{\rho}{d_2}\right)A\right) , \tag{44}$$

where ${}_1F_1(a;b;c)$ is the degenerate hypergeometric function [43]. For $\rho = 1$ $(\Gamma = \Delta = 0)$ A and C are independent and have the joint probability of

$$P(A,C) = \frac{4A}{\langle I\rangle^2}\exp\left(-\frac{2A}{\langle I\rangle}\right)\frac{4C}{\langle I\rangle^2}\exp\left(-\frac{2C}{\langle I\rangle}\right) . \tag{45}$$

The PDF of B was found only for $\rho = 1$ $(\Gamma = \Delta = 0)$

$$P(B) = \frac{1}{\langle I\rangle^2}\exp\left(-\frac{2|B|}{\langle I\rangle}\right)\left(\frac{\langle I\rangle}{2}+|B|\right) . \tag{46}$$

B is highly correlated to A and C even for $\rho = 1$.

Because B includes multiples [Eq. (40b)] of independent Gaussian variables, its odd moments vanish, meaning that for arbitrary ρ its PDF is symmetric around zero.

For $\rho = 0$, $\Gamma = 1$, and $\Delta = 0$ (no scattering media at all) $P(B) = \delta(B)$ and $A = C = I_i$, meaning that the measured intensity is constant [Eq. (38)] and equal to the incident one (I_i).

V. SUMMARY

In this paper the vector statistics of partially polarized light of spatially or temporally randomness was investigated assuming Gaussian correlated fields. Statistics were found for the ellipse variables which depend on the 2×2 covariance Hermitian matrix. The real part of the covariance matrix crossed term is responsible for breaking the symmetry of the PDF of $\psi_\pm$ and U around

$\psi_{\pm} = 0$ and $U = 0$, respectively, while the imaginary part breaks the symmetry of $P(V)$ around $V = 0$. The PDF of the four Stokes variables was found also, where the last three of them Q, U, and V have a universal probability function form.

The explicit form of the covariance matrix for scattered wave from random media (speckle patterns) with slab geometry and for different polarization state of the input beam was given.

In general the statistics depend on four parameters. For linearly (or circularly) input polarization, the statistics were found to be two-parameter dependent, where one is connected to the input beam intensity ($\langle I \rangle$) and the second is P the degree of polarization. In case of elliptic input polarization state the system needs three parameters to describe the statistics, while the degree of polarization depends also on the input minor to major polarization field ratio.

The rotator of the input linearly polarization direction was also investigated, the statistics defined by an extra parameters (Γ or Δ) beside ρ, as was predicted experimentally [37]. Still, in this problem there are many open questions, caused by mathematical difficulties, which remain to be solved. These statistics offer possibilities for investigating the polarization of multiple-scattering processes from random media.

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